

Determining White Noise Forcing From Eulerian Observations in the Navier Stokes Equation

Viet Ha Hoang · Kody J. H. Law ·
Andrew M. Stuart

the date of receipt and acceptance should be inserted later

Abstract The Bayesian approach to inverse problems is of paramount importance in quantifying uncertainty about the input to and the state of a system of interest given noisy observations. Herein we consider the forward problem of the forced 2D Navier Stokes equation. The inverse problem is inference of the forcing, and possibly the initial condition, given noisy observations of the velocity field. We place a prior on the forcing which is in the form of a spatially correlated temporally white Gaussian process, and formulate the inverse problem for the posterior distribution. Given appropriate spatial regularity conditions, we show that the solution is a continuous function of the forcing. Hence, for appropriately chosen spatial regularity in the prior, the posterior distribution on the forcing is absolutely continuous with respect to the prior and is hence well-defined. Furthermore, the posterior distribution is a continuous function of the data. We complement this theoretical result with numerical simulation of the posterior distribution.

1 Introduction

The Bayesian approach to inverse problems has grown in popularity significantly over the last decade, driven by algorithmic innovation and steadily increasing computer power [7]. Recently there have been systematic developments of the theory of Bayesian inversion on function space [8, 9, 2, 15, 10, 11]

Viet Ha Hoang
Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, 637371. E-mail: vhhoang@ntu.edu.sg

Kody J. H. Law
Mathematics Institute, Warwick University, Coventry CV4 7AL, UK. E-mail: k.j.h.law@warwick.ac.uk

Andrew M. Stuart
Mathematics Institute, Warwick University, Coventry CV4 7AL, UK. E-mail: a.m.stuart@warwick.ac.uk

and this has led to new sampling algorithms which perform well under mesh-refinement [1, 17]. In this paper we add to this growing interest in the Bayesian formulation of inversion, in the context of a specific PDE inverse problem, motivated by geophysical applications such as atmospheric/oceanographic data assimilation, and demonstrate that fully Bayesian probing of the posterior distribution is feasible.

The primary goal of the paper is to demonstrate that the Bayesian formulation of inversion for the forced Navier-Stokes equation, introduced in [2], can be extended to the case of white noise forcing; the paper [2] assumed an Ornstein-Uhlenbeck structure in time for the forcing, and hence did not include the white noise case. It is technically demanding to extend to the case of white noise forcing, but it is also of practical interest. This practical importance stems from the fact that the Bayesian formulation of problems with white noise forcing corresponds to a statistical version of the continuous time weak constraint 4DVAR methodology [18]. The 4DVAR approach to data assimilation currently gives the most accurate global short term weather forecasts available [13] and this is arguably the case because, unlike ensemble filters which form the major competitor, 4DVAR has a rigorous statistical interpretation as a MAP estimator. It is therefore of interest to seek to embed our understanding of such methods in a broader Bayesian context.

The key tools required in applying the function space Bayesian approach in [15] are the proof of continuity of the forward map from the function space of the unknowns to the data space, together with estimates of the dependence of the forward map upon its point of application, sufficient to show certain integrability properties with respect to the prior. This program is carried out for the 2D Navier-Stokes equation with Ornstein-Uhlenbeck priors on the forcing in the paper [2]. However to use priors which are white in time adds further complications since it is necessary to study the stochastically forced 2D Navier-Stokes equation and to establish continuity of the solution with respect to small changes in the Brownian motion which defines the stochastic forcing. We do this by employing the solution concept introduced by Flandoli in [4], and using probabilistic estimates on the solution derived by Mattingly in [14]. In section 2 we describe the relevant theory of the forward problem, employing the setting of Flandoli. In section 3 we build on this theory, using the estimates of Mattingly to verify the conditions in [15], resulting in a well-posed Bayesian inverse problem for which the posterior is Lipschitz in the data with respect to Hellinger metric. Section 4 extends this to include making inference about the initial condition as well as the forcing. Finally, in section 5, we present numerical results which demonstrate feasibility of sampling from the posterior on white noise forces.

2 Forward Problem

In this section we study the forward problem of the Navier-Stokes equation driven by white noise. Subsection 2.1 describes the forward problem, the

Navier-Stokes equation, and rewrites it as an ordinary differential equation in a Hilbert space. In subsection 2.2 we define the functional setting used throughout the paper. Subsection 2.3 highlights the solution concept that we use, leading in subsection 2.4 to proof of the key fact that the solution of the Navier-Stokes equation is continuous as a function of the rough driving of interest.

2.1 Overview

Let $D \in \mathbb{R}^2$ be a bounded domain with smooth boundary. We consider in D the Navier-Stokes equation

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u &= f - \nabla p, & (x, t) \in D \times (0, \infty) \\ \nabla \cdot u &= 0, & (x, t) \in D \times (0, \infty) \\ u &= 0, & (x, t) \in \partial D \times (0, \infty), \\ u &= u_0, & (x, t) \in D \times \{0\}. \end{aligned} \quad (1)$$

We assume that the initial condition u_0 and the forcing $f(\cdot, t)$ are divergence-free. We denote by \mathbf{V} the space of all divergence-free smooth functions from D to \mathbb{R}^2 with compact support, by \mathbb{H} the closure of \mathbf{V} in $(L^2(D))^2$, and by \mathbb{H}^1 the closure of \mathbf{V} in $(H^1(D))^2$. Let $\mathbb{H}^2 = (H^2(D))^2 \cap \mathbb{H}^1$. The initial condition u_0 is assumed to be in \mathbb{H} . We define the linear Stokes' operator $A : \mathbb{H}^2 \rightarrow \mathbb{H}$ by $Au = -\Delta u$ noting that the assumption of compact support means that Dirichlet boundary condition are imposed on the Stokes' operator A .¹ Since A is selfadjoint, A possesses eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with the corresponding eigenvectors $e_1, e_2, \dots \in \mathbb{H}^2$.

We denote by $\langle \cdot, \cdot \rangle$ the inner product in \mathbb{H} , and extend to a dual pairing on $\mathbb{H}^{-1} \times \mathbb{H}^1$. We then define the bilinear form $B : \mathbb{H}^1 \times \mathbb{H}^1 \rightarrow \mathbb{H}^{-1}$

$$\langle B(u, v), z \rangle = \int_D z(x) \cdot (u(x) \cdot \nabla) v(x) dx$$

which must hold for all $z \in \mathbb{H}^1$. From the incompressibility condition we have, for all $z \in \mathbb{H}^1$,

$$\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle. \quad (2)$$

By projecting problem (1) into \mathbb{H} we may write it as an ordinary differential equation in the form

$$du(t) = -\nu A u dt - B(u, u) dt + dW(t), \quad u(0) = u_0 \in \mathbb{H}, \quad (3)$$

where $dW(t)$ is the projection of the forcing $f(x, t)dt$ into \mathbb{H} . We will define the solution of this equation pathwise, for suitable W , not necessarily differentiable in time.

¹ If, instead, we consider the problem on the torus \mathbb{T}^2 , all the results below hold.

2.2 Function Spaces

For any $s \geq 0$ we define $\mathbb{H}^s \subset \mathbb{H}$ to be the Hilbert space of functions $u = \sum_{k=1}^{\infty} u_k e_k \in \mathbb{H}$ such that

$$\sum_{k=1}^{\infty} \lambda_k^s u_k^2 < \infty;$$

we note that the \mathbb{H}^j for $j \in \{0, 1, 2\}$ coincide with the preceding definitions of these spaces. The space \mathbb{H}^s is endowed with the inner product

$$\langle u, v \rangle_{\mathbb{H}^s} = \sum_{k=1}^{\infty} \lambda_k^s u_k v_k,$$

for $u = \sum_{k=1}^{\infty} u_k e_k, v = \sum_{k=1}^{\infty} v_k e_k \in \mathbb{H}$. We denote by \mathbb{V} the particular choice $s = \frac{1}{2} + \epsilon$, namely $\mathbb{H}^{\frac{1}{2} + \epsilon}$, for $\epsilon > 0$. In what follows we will be particularly interested in continuity of the mapping from the forcing W into linear functionals of the solution of (3). To this end it is helpful to define the Banach space $\mathbb{X} := C([0, T]; \mathbb{V})$ with the norm

$$\|W\|_{\mathbb{X}} = \sup_{t \in (0, T)} \|W(t)\|_{\mathbb{V}}.$$

2.3 Solution Concept

In what follows we define a solution concept for equation (3) in the case where W is continuous, but not necessarily differentiable, in time. We always assume that $W(0) = 0$. The approach is based on that in Flandoli [4]. In this approach we first make sense of the equation

$$dz(t) = -\nu A z dt + dW(t), \quad z(0) = 0 \in \mathbb{H}. \quad (4)$$

Then we write $u = z + v$ and notice that v solves the equation

$$dv(t) = -\nu A v dt - B(z + v, z + v) dt, \quad v(0) = u_0 \in \mathbb{H}. \quad (5)$$

We make sense of this equation for v and then deduce that $u = z + v$ is a weak solution (defined below) of (3). When we wish to emphasize the dependence of u on W (and similarly for z and v) we write $u(t; W)$.

For each $W \in \mathbb{X}$, we define the weak solution $u(\cdot; W) \in C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^{1/2})$ of (3) as a function that satisfies

$$\langle u(t), \phi \rangle + \nu \int_0^t \langle u(s), A \phi \rangle ds - \int_0^t \langle B(u(s), \phi), u(s) \rangle ds = \langle u_0, \phi \rangle + \langle W(t), \phi \rangle, \quad (6)$$

for all $\phi \in \mathbb{H}^2$ and all $t \in (0, T)$; note the integration by parts on the Stokes' operator and the use of (2) to derive this identity from (3).

Application of the variation of constants formula to (4), together with an integration by parts, yields

$$\begin{aligned} z(t) &= \int_0^t e^{-\nu A(t-s)} dW(s) \\ &= W(t) - \int_0^t \nu A e^{-\nu A(t-s)} W(s) ds. \end{aligned}$$

We now justify this formula for z . We define $w_k = \langle W, e_k \rangle$, that is

$$W(t) := \sum_{k=1}^{\infty} w_k(t) e_k \in \mathbb{X}. \quad (7)$$

We then define

$$z(t; W) = W(t) - \sum_{k=1}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k, \quad (8)$$

noting that this agrees with the expression for $z(t)$ above. We then have the following:

Lemma 1 *For each $W \in \mathbb{X}$, the function $z = z(\cdot; W) \in C([0, T]; \mathbb{H}^{1/2})$.*

Proof We first show that for each t , $z(t; W)$ as defined in (8) belongs to $\mathbb{H}^{1/2}$. Fixing an integer $M > 0$, using inequality $a^{1-\epsilon/2} e^{-a} < c$ for all $a > 0$ for an appropriate constant c , we have

$$\sum_{k=1}^M \lambda_k^{1/2} \left(\int_0^t \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} w_k(s) ds \right)^2 \leq \sum_{k=1}^M \lambda_k^{1/2} \left(\int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \lambda_k^{\epsilon/2} |w_k(s)| dx \right)^2.$$

Therefore,

$$\begin{aligned} \left\| \sum_{k=1}^M \int_0^t \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} w_k(s) e_k ds \right\|_{\mathbb{H}^{1/2}} &\leq \left\| \sum_{k=1}^M \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \lambda_k^{\epsilon/2} |w_k(s)| e_k ds \right\|_{\mathbb{H}^{1/2}} \\ &\leq \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \left\| \sum_{k=1}^M \lambda_k^{\epsilon/2} |w_k(s)| e_k \right\|_{\mathbb{H}^{1/2}} ds \\ &\leq \max_{s \in (0, T)} \|W(s)\|_{\mathbb{H}^{1/2+\epsilon}} \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} ds, \end{aligned}$$

which is uniformly bounded for all M . Therefore,

$$\sum_{k=1}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \in \mathbb{H}^{1/2},$$

It follows from (8) that, since $W \in \mathbb{X}$, for each t , $z(t; W) \in \mathbb{H}^{1/2}$ as required. Furthermore, for all $t \in (0, T)$

$$\|z(t; W)\|_{\mathbb{H}^{1/2}} \leq c \|W\|_{\mathbb{X}}. \quad (9)$$

Now we turn to the continuity in time. Arguing similarly, we have that

$$\begin{aligned} \left\| \sum_{k=M}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} &\leq \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \left\| \sum_{k=M}^{\infty} w_k(s) e_k \right\|_{\mathbb{H}^{1/2+\epsilon}} ds \\ &\leq \left(\int_0^t \frac{c}{(t-s)^{(1-\epsilon/2)p}} ds \right)^{1/p} \left(\int_0^t \left\| \sum_{k=M}^{\infty} w_k(s) e_k \right\|_{\mathbb{H}^{1/2+\epsilon}}^q ds \right)^{1/q}, \end{aligned}$$

for all $p, q > 0$ such that $1/p + 1/q = 1$. From the Lebesgue dominated convergence theorem,

$$\lim_{M \rightarrow \infty} \int_0^t \left\| \sum_{k=M}^{\infty} w_k(s) e_k \right\|_{\mathbb{H}^{1/2+\epsilon}}^q ds = 0;$$

and when p sufficiently close to 1,

$$\int_0^t \frac{c}{(t-s)^{(1-\epsilon/2)p}} ds$$

is finite. We then deduce that

$$\lim_{M \rightarrow \infty} \left\| \sum_{k=M}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}}^{1/2} = 0,$$

uniformly for all t .

Fixing $t \in (0, T)$ we show that

$$\lim_{t' \rightarrow t} \|z(t; W) - z(t'; W)\|_{\mathbb{H}^{1/2}} = 0.$$

We have

$$\begin{aligned} \|z(t; W) - z(t'; W)\|_{\mathbb{H}^{1/2}} &\leq \|W(t) - W(t')\|_{\mathbb{H}^{1/2}} + \\ &\left\| \sum_{k=1}^{M-1} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds - \int_0^{t'} w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} + \\ &\left\| \sum_{k=M}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} + \left\| \sum_{k=M}^{\infty} \left(\int_0^{t'} w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}}. \end{aligned}$$

For $\delta > 0$, when M is sufficiently large, the argument above shows that

$$\left\| \sum_{k=M}^{\infty} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} + \left\| \sum_{k=M}^{\infty} \left(\int_0^{t'} w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} < \delta/3.$$

Furthermore, when $|t' - t|$ is sufficiently small,

$$\left\| \sum_{k=1}^{M-1} \left(\int_0^t w_k(s) \nu \lambda_k e^{(t-s)(-\nu \lambda_k)} ds - \int_0^{t'} w_k(s) \nu \lambda_k e^{(t'-s)(-\nu \lambda_k)} ds \right) e_k \right\|_{\mathbb{H}^{1/2}} < \delta/3.$$

Finally, since $W \in \mathbb{X}$, for $|t' - t|$ is sufficiently small we have

$$\|W(t) - W(t')\|_{\mathbb{H}^{1/2}} < \delta/3$$

Thus when $|t' - t|$ is sufficiently small, $\|z(t; W) - z(t'; W)\|_{\mathbb{H}^{1/2}} < \delta$. The conclusion follows. \square

Having established regularity, we now show that z is indeed a weak solution of (4).

Lemma 2 *For each $\phi \in \mathbb{H}^2$, $z(t) = z(t; W)$ satisfies*

$$\langle z(t), \phi \rangle + \nu \int_0^t \langle z(s), A\phi \rangle ds = \langle W(t), \phi \rangle.$$

Proof It is sufficient to show this for $\phi = e_k$. We have

$$\begin{aligned} \int_0^t \langle z(s), Ae_k \rangle ds &= \int_0^t \langle W(s), Ae_k \rangle ds - \int_0^t \int_0^s w_k(\tau) \nu \lambda_k^2 e^{(s-\tau)(-\nu\lambda_k)} d\tau ds \\ &= \lambda_k \int_0^t w_k(s) ds - \nu \lambda_k^2 \int_0^t w_k(\tau) \left(\int_\tau^t e^{(s-\tau)(-\nu\lambda_k)} ds \right) d\tau \\ &= \lambda_k \int_0^t w_k(s) ds - \lambda_k \int_0^t w_k(\tau) d\tau + \lambda_k \int_0^t w_k(\tau) e^{(t-\tau)(-\nu\lambda_k)} d\tau \\ &= \lambda_k \int_0^t w_k(\tau) e^{(t-\tau)(-\nu\lambda_k)} d\tau. \end{aligned}$$

On the other hand,

$$\langle z(t), e_k \rangle = \langle W(t), e_k \rangle - \nu \lambda_k \int_0^t w_k(s) e^{(t-s)(-\nu\lambda_k)} ds.$$

The result then follows. \square

We now turn to the following result, which concerns v and is established on page 416 of [4], given the properties of $z(\cdot; W)$ established in the preceding two lemmas.

Lemma 3 *For each $W \in \mathbb{X}$, problem (5) has a unique solution $v \in C(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^1)$.*

We then have the following existence and uniqueness result for the Navier-Stokes equation (3), more precisely for the weak form (6), driven by rough additive forcing [4]:

Proposition 1 *For each $W \in \mathbb{X}$, problem (6) has a unique solution $u \in C(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^{1/2})$ such that $u - z \in L^2(0, T; \mathbb{H}^1)$.*

Proof A solution u for (6) can be taken as

$$u(t; W) = z(t; W) + v(t; W). \quad (10)$$

Assume that $\bar{u}(t; W)$ is another solution of (6). Then $\bar{v}(t; W) = \bar{u}(t; W) - z(t; W)$ is a solution in $C(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^1)$ of (5). However, (5) has a unique solution in $C(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^1)$. Thus $\bar{v} = v$. \square

2.4 Continuity of the Forward Map

The purpose of this subsection is to establish continuity of the forward map from W into the weak solution u of (3), as defined in (6), at time $t > 0$.

Theorem 1 *For each $t > 0$, the solution $u(t; \cdot)$ of (3) is a continuous map from \mathbb{X} into \mathbb{H} .*

Proof We consider equation (3) with driving $W \in \mathbb{X}$ given by (7) and by $W' \in \mathbb{X}$ defined by

$$W'(s) = \sum_{k=1}^{\infty} w_k'(s) e_k \in \mathbb{X}.$$

We will prove that, for W, W' from a bounded set in \mathbb{X} , there is $c = C(T) > 0$, such that

$$\sup_{t \in (0, T)} \|z(t; W) - z(t; W')\|_{\mathbb{H}^{1/2}} \leq c \|W - W'\|_{\mathbb{X}} \quad (11)$$

and, for each $t \in (0, T)$,

$$\|v(t; W) - v(t; W')\|_{\mathbb{H}}^2 \leq c \sup_{s \in (0, T)} \|z(s; W) - z(s; W')\|_{L^4(D)}^2. \quad (12)$$

This suffices to prove the desired result since Sobolev embedding yields, from (12),

$$\|v(t; W) - v(t; W')\|_{\mathbb{H}}^2 \leq c \sup_{s \in (0, T)} \|z(s; W) - z(s; W')\|_{\mathbb{H}^{\frac{1}{2}}}^2. \quad (13)$$

Since $u = z + v$ we deduce from (11) and (13) that u as a map from \mathbb{X} to \mathbb{H} is continuous.

To prove (11) we note that

$$\|z(t; W) - z(t; W')\|_{\mathbb{H}^{\frac{1}{2}}} \leq \|W(t) - W'(t)\|_{\mathbb{H}^{\frac{1}{2}}} + \left\| \int_0^t \nu A e^{-\nu A(t-s)} (W(s) - W'(s)) ds \right\|_{\mathbb{H}^{\frac{1}{2}}}$$

so that

$$\begin{aligned} \sup_{t \in (0, T)} \|z(t; W) - z(t; W')\|_{\mathbb{H}^{\frac{1}{2}}} &\leq \|W - W'\|_{\mathbb{X}} \\ &+ \sup_{t \in (0, T)} \left\| \int_0^t \nu A e^{-\nu A(t-s)} (W(s) - W'(s)) ds \right\|_{\mathbb{H}^{\frac{1}{2}}}. \end{aligned}$$

Thus it suffices to consider the last term on the right hand side. We have

$$\begin{aligned} &\left\| \int_0^t A e^{-\nu A(t-s)} (W(s) - W'(s)) ds \right\|_{\mathbb{H}^{\frac{1}{2}}}^2 \\ &= \left\| \sum_{k=1}^{\infty} \int_0^t \lambda_k e^{(t-s)(-\nu \lambda_k)} (w_k'(s) - w_k(s)) e_k ds \right\|_{\mathbb{H}^{1/2}}^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \lambda_k^{1/2} \left(\int_0^t \lambda_k e^{(t-s)(-\nu\lambda_k)} (w'_k(s) - w_k(s)) ds \right)^2 \\
&\leq \sum_{k=1}^{\infty} \lambda_k^{1/2} \left(\int_0^t \lambda_k e^{(t-s)(-\nu\lambda_k)} |w'_k(s) - w_k(s)| ds \right)^2 \\
&\leq \sum_{k=1}^{\infty} \lambda_k^{1/2} \left(\int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \lambda_k^{\epsilon/2} |w'_k(s) - w_k(s)| ds \right)^2
\end{aligned}$$

where we have used the fact that $a^{1-\epsilon/2}e^{-a} < c$ for all $a > 0$ for an appropriate constant c . From this, we deduce that

$$\begin{aligned}
&\left\| \int_0^t A e^{-\nu A(t-s)} (W(s) - W'(s)) ds \right\|_{\mathbb{H}^{\frac{1}{2}}} \\
&\leq \left\| \sum_{k=1}^{\infty} \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \lambda_k^{\epsilon/2} |w'_k(s) - w_k(s)| e_k ds \right\|_{\mathbb{H}^{1/2}} \\
&\leq \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} \left\| \sum_{k=1}^{\infty} \lambda_k^{\epsilon/2} |w'_k(s) - w_k(s)| e_k \right\|_{\mathbb{H}^{1/2}} ds \\
&\leq \int_0^t \frac{c}{(t-s)^{1-\epsilon/2}} ds \sup_{s \in (0, T)} \left\| \sum_{k=1}^{\infty} \lambda_k^{\epsilon/2} |w'_k(s) - w_k(s)| e_k \right\|_{\mathbb{H}^{1/2}} \\
&\leq c \sup_{s \in (0, T)} \|W'(s) - W(s)\|_{\mathbb{V}}.
\end{aligned}$$

Therefore (11) holds.

We now prove (12). We will use the following estimate for the solution v of (5) which is proved in Flandoli [4], page 412, by means of a Gronwall argument:

$$\sup_{s \in (0, T)} \|v(s)\|_{\mathbb{H}} + \int_0^T \|v(s)\|_{\mathbb{H}^1}^2 ds \leq C(T, \sup_{s \in (0, T)} \|z(s)\|_{L^4(D)}). \quad (14)$$

We show that the map $C([0, T]; L^4(D)) \ni z(\cdot; W) \mapsto v(\cdot; W) \in \mathbb{H}$ is continuous. For W and W' in \mathbb{X} , define $v = v(t; W)$, $v' = v(t; W')$, $z = z(t; W)$, $z' = z(t; W')$, $e = v - v'$ and $\delta = z - z'$. Then we have

$$\frac{de}{dt} + \nu A e + B(v + z, v + z) - B(v' + z', v' + z') = 0. \quad (15)$$

From this, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d\|e\|_{\mathbb{H}}^2}{dt} + \nu \|e\|_{\mathbb{H}^1}^2 = \\
&\quad -\langle B(v + z, v + z), e \rangle \\
&\quad + \langle B(v' + z', v' + z'), e \rangle.
\end{aligned}$$

From (2) we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d\|e\|_{\mathbb{H}}^2}{dt} + \nu \|e\|_{\mathbb{H}^1}^2 = \\
& \quad + \langle B(v+z, e), v+z \rangle \\
& \quad - \langle B(v'+z', e), v'+z' \rangle \\
& = \langle B(v+z, e), v+z-v'-z' \rangle \\
& \quad - \langle B(v'+z'-v-z, e), v'+z' \rangle \\
& = \langle B(v+z, e), e+\delta \rangle \\
& \quad + \langle B(e+\delta, e), v'+z' \rangle \\
& \leq (\|e\|_{L^4(D)} + \|\delta\|_{L^4(D)}) (\|v\|_{L^4(D)} + \|z\|_{L^4(D)} + \|v'\|_{L^4(D)} + \|z'\|_{L^4(D)}) \|e\|_{\mathbb{H}^1}.
\end{aligned}$$

Using the interpolation inequality

$$\|e\|_{L^4(D)} \leq c \|e\|_{\mathbb{H}^1}^{1/2} \|e\|_{\mathbb{H}}^{1/2},$$

we get

$$\begin{aligned}
& \frac{1}{2} \frac{d\|e\|_{\mathbb{H}}^2}{dt} + \nu \|e\|_{\mathbb{H}^1}^2 \leq \\
& \quad c \left(\|e\|_{\mathbb{H}^1}^{3/2} \|e\|_{\mathbb{H}}^{1/2} + \|\delta\|_{L^4(D)} \|e\|_{\mathbb{H}^1} \right) \cdot \left(\|v\|_{L^4(D)} + \|v'\|_{L^4(D)} + \|z\|_{L^4(D)} + \|z'\|_{L^4(D)} \right).
\end{aligned}$$

From the Young inequality, we have

$$\frac{1}{2} \frac{d\|e\|_{\mathbb{H}}^2}{dt} + \nu \|e\|_{\mathbb{H}^1}^2 \leq \nu \|e\|_{\mathbb{H}^1}^2 + c \|e\|_{\mathbb{H}}^2 \cdot I_4 + c \|\delta\|_{L^4(D)}^2 \cdot I_2. \quad (16)$$

where we have defined

$$\begin{aligned}
I_2 &= \|v\|_{L^4(D)}^2 + \|v'\|_{L^4(D)}^2 + \|z\|_{L^4(D)}^2 + \|z'\|_{L^4(D)}^2 \\
I_4 &= \|v\|_{L^4(D)}^4 + \|v'\|_{L^4(D)}^4 + \|z\|_{L^4(D)}^4 + \|z'\|_{L^4(D)}^4.
\end{aligned}$$

From Gronwall's inequality, we have

$$\|e(t)\|_{\mathbb{H}}^2 \leq c \int_0^t \left(e^{\int_s^t I_4(s') ds'} \right) \|\delta(s)\|_{L^4(D)}^2 I_2(s) ds. \quad (17)$$

From the interpolation inequality

$$\|v(s'; W)\|_{L^4(D)} \leq c \|v(s'; W)\|_{\mathbb{H}^1}^{1/2} \|v(s'; W)\|_{\mathbb{H}}^{1/2},$$

we have that

$$\int_0^T \|v(s'; W)\|_{L^4(D)}^4 ds' \leq c \sup_{s' \in (0, T)} \|v(s'; W)\|_{\mathbb{H}}^2 \int_0^T \|v(s'; W)\|_{\mathbb{H}^1}^2 ds',$$

which is bounded uniformly when W belongs to a bounded subset of \mathbb{X} due to (14) and (9). Therefore

$$\|e(t)\|_{\mathbb{H}}^2 \leq c \sup_{0 \leq s \leq T} \|\delta(s)\|_{L^4(D)}^2.$$

□

3 Bayesian Inverse Problems With Model Error

In this section we formulate the inverse problem of determining the forcing to equation (3) from knowledge of the velocity field; more specifically we formulate the Bayesian inverse problem of determining the driving Brownian motion W from noisy pointwise observations of the velocity field.

We set-up the likelihood in subsection 3.1. Then, in subsection 3.2, we describe the prior on the forcing which is a Gaussian white-in-time process with spatial correlations, and hence a spatially correlated Brownian motion prior on W . This leads, in subsection 3.3, to a well-defined posterior distribution, absolutely continuous with respect to the prior, and Lipschitz in the Hellinger metric with respect to the data.

3.1 Likelihood

Fix a collection of times $t_j \in (0, T)$, $j = 1, \dots, J$. Let ℓ be a collection of K linear functionals on \mathbb{H} . We assume that we observe, for each j , $\ell(u(\cdot, t_j; W))$ plus a draw from a centered Gaussian noise ϑ with variance σ , i.e.

$$\delta_j = \ell(u(\cdot, t_j; W)) + \vartheta_j, \quad (18)$$

is known to us. Concatenating the data we obtain

$$\delta = \mathcal{G}(W) + \vartheta \quad (19)$$

where $\delta, \vartheta \in \mathbb{R}^{JK}$ and $\mathcal{G} : \mathbb{X} \rightarrow \mathbb{R}^{JK}$. We assume that the observational noise ϑ is a draw from the Gaussian $N(0, \Sigma)$ on \mathbb{R}^{JK} .

In the following we will define a prior measure ρ on W and then determine the conditional probability measure $\rho^\delta = \mathbb{P}(W|\delta)$. We will then show that ρ^δ is absolutely continuous with respect to ρ and that the Radon-Nikodym derivative between the measures is given by

$$\frac{d\rho^\delta}{d\rho} \propto \exp\left(-\Phi(W; \delta)\right), \quad (20)$$

where

$$\Phi(W; \delta) = \frac{1}{2} \left| \Sigma^{-\frac{1}{2}} (\delta - \mathcal{G}(W)) \right|^2. \quad (21)$$

The right hand side of (20) is the likelihood of the data δ .

3.2 Prior

We construct our prior on the time-integral of the forcing, namely W . Let Q be a linear operator from the Hilbert space $\mathbb{H}^{\frac{1}{2}+\epsilon}$ into itself with eigenvectors e_k and eigenvalues σ_k^2 for $k = 1, 2, \dots$. We make the following assumption

Assumption 1 *There is an $\epsilon > 0$ such that the coefficients $\{\sigma_k\}$ satisfy*

$$\sum_{k=1}^{\infty} \sigma_k^2 \lambda_k^{1/2+\epsilon} < \infty.$$

As

$$\sum_{k=1}^{\infty} \langle Q e_k, e_k \rangle_{\mathbb{H}^{\frac{1}{2}+\epsilon}} = \sum_{k=1}^{\infty} \sigma_k^2 \lambda_k^{\frac{1}{2}+\epsilon} < \infty,$$

Q is a trace class operator in $\mathbb{H}^{\frac{1}{2}+\epsilon}$.

We assume that our prior is the Q -Wiener process W with values in $\mathbb{H}^{\frac{1}{2}+\epsilon}$ where $W(s_1) - W(s_2)$ is Gaussian in $\mathbb{H}^{\frac{1}{2}+\epsilon}$ with covariance $(s_1 - s_2)Q$ and mean 0. In the mean square norm, this process can be written as

$$W(t) = \sum_{k=1}^{\infty} \sigma_k e_k \beta_k(t), \quad (22)$$

where $\beta_k(t)$ are pair-wise independent Brownian motions (see Da Prato and Zabczyk [3], Chapter 4). We define by ρ the measure generated by this Q -Wiener process on \mathbb{X} .

Remark 1 We now show that under Assumption 1, ρ almost surely, solution u of (6) defined in (10) equals the unique progressively measurable solution in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^1)$ constructed in Flandoli [4] when the noise W is sufficiently spatially regular. This enables to employ the boundedness of the second moment of the energy $\mathbb{E}^\rho[\|u(\cdot, t; W)\|_{\mathbb{H}}^2]$, established in Mattingly [14], which we need later.

For the infinite dimensional Brownian motion W defined in (22) where

$$\sum_{k=1}^{\infty} \lambda_k^{2\beta_0-1/2} \sigma_k^2 < \infty,$$

for some $\beta_0 > 0$, Flandoli [4] employs the Ornstein-Uhlenbeck process

$$z_\alpha(t) = \int_{-\infty}^t e^{-(\nu A + \alpha)(t-s)} dW(s) \quad (23)$$

where α is a constant in order to define a solution of (6). Note that if $\beta_0 > \frac{1}{2}$ then Assumption 1 is satisfied. With respect to the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, the expectation $\mathbb{E}\|z_\alpha(t)\|_{\mathbb{H}^{1/2+2\beta}}^2$ is finite for $\beta < \beta_0$. Thus almost surely with respect to $(\Omega, \mathcal{F}_t, \mathbb{P})$, $z_\alpha(t)$ is sufficiently regular so that problem (5) with the initial condition $v(0; W) = u_0 - z_\alpha(0)$ is well posed. The stochastic solution to the problem (1) is defined as

$$u(\cdot, t; W) = z_\alpha(t; W) + v(t; W). \quad (24)$$

When $\beta_0 > \frac{1}{2}$, $\mathbb{E}\|z_\alpha(t)\|_{\mathbb{H}^1}^2$ is finite so $u(\cdot, t; W) \in C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^1)$. Flandoli [4] leaves open the question of the uniqueness of a generalized solution

to (6) in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^{1/2})$. However, there is a unique solution in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^1)$.

Almost surely with respect to the probability measure ρ , solution u of (6) constructed in (10) equals the solution constructed by Flandoli [4] in (24). To see this, note that the stochastic integral

$$\int_0^t e^{-\nu A(t-s)} dW(s) \quad (25)$$

can be written in the integration by parts form (8). Therefore, with respect to ρ ,

$$\begin{aligned} \mathbb{E}^\rho[\|z(t)\|_{L^2(0, T; \mathbb{H}^1)}^2] &= \int_0^T \mathbb{E}^\rho\left[\sum_{k=1}^\infty \lambda_k \sigma_k^2 \left(\int_0^t e^{-\nu \lambda_k(t-s)} d\beta_k(s)\right)^2\right] dt \\ &= \int_0^T \left(\sum_{k=1}^\infty \lambda_k \sigma_k^2 \int_0^t e^{-2\nu \lambda_k(t-s)} ds\right) dt = \frac{1}{2\nu} \int_0^T \sum_{k=1}^\infty \sigma_k^2 (1 - e^{-2\nu \lambda_k t}) dt \end{aligned}$$

which is finite. Therefore ρ almost surely, $z(t) \in L^2(0, T; \mathbb{H}^1)$. Thus $u(t; W) \in C(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^1)$. We can then argue that ρ almost surely, the solution u constructed in (10) equals Flandoli's solution in (24) which we denote by u_α (even though it does not depend on α) as follows. As $u_\alpha \in C([0, T]; H) \cap L^2([0, T]; \mathbb{H}^1)$, $v_\alpha(t; W) = u_\alpha(t; W) - z(t; W) \in C([0, T]; H) \cap L^2([0, T]; \mathbb{H}^1)$ and satisfies (5). As for each W , (5) has a unique solution in $C([0, T]; H) \cap L^2([0, T]; \mathbb{H}^1)$, so $v_\alpha(t; W) = v(t; W)$. Thus almost surely, the Flandoli [4] solution equals the solution u in (10). This is also the argument to show that (3) has a unique solution in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^1)$.

3.3 Posterior

Theorem 2 *The conditional measure $\mathbb{P}(W|\delta) = \rho^\delta$ is absolutely continuous with respect to the prior measure ρ with the Radon-Nikodym derivative being given by (20). Furthermore, there is a constant c so that*

$$d_{\text{Hell}}(\rho^\delta, \rho^{\delta'}) \leq c|\delta - \delta'|.$$

Proof From Corollary 2.1 of Cotter et al. [2], it suffices to show that the mapping $\mathcal{G} : \mathbb{X} \rightarrow \mathbb{R}^{JK}$ is continuous, in order to deduce that $\rho^\delta \ll \rho$ and that the Radon-Nikodym derivative (20) holds. This follows from Theorem 1. The Lipschitz continuity of the posterior in the Hellinger metric follows similarly to the proof of Theorem 2.5 in [2] and we only sketch the proof. As in that proof we observe that it is sufficient to consider observations at one time $t_0 > 0$. We define

$$Z(\delta) := \int_{\mathbb{X}} \exp(-\Phi(W; \delta)) d\rho(W).$$

Mattingly [14] shows that for each t , $\mathbb{E}^\rho(\|u(\cdot, t; W)\|_{\mathbb{H}}^2)$ is bounded. Fixing a large constant M , the probability that $\|u(\cdot, t; W)\|_{\mathbb{H}} \leq M$ is larger than $1 - c/M > 1/2$. For such a path W ,

$$\Phi(W; \delta) \leq c(|\delta| + \|\ell\|_{\mathbb{H}^{-1}} M).$$

From this, we deduce that $Z(\delta) > 0$. Next, we have that

$$\begin{aligned} |Z(\delta) - Z(\delta')| &\leq \int_{\mathbb{X}} |\Phi(W; \delta) - \Phi(W; \delta')| d\rho(W) \\ &\leq c \int_{\mathbb{X}} (|\delta| + |\delta'| + 2|\ell(u(t_0; W))|) |\delta - \delta'| d\rho(W) \\ &\leq c|\delta - \delta'|. \end{aligned}$$

We then have

$$\begin{aligned} 2d_{\text{Hell}}(\rho^\delta, \rho^{\delta'})^2 &\leq \int_{\mathbb{X}} \left(Z(\delta)^{-1/2} \exp(-\frac{1}{2}\Phi(W; \delta)) - Z(\delta')^{-1/2} \exp(-\frac{1}{2}\Phi(W; \delta')) \right)^2 d\rho(W) \\ &\leq I_1 + I_2, \end{aligned}$$

where

$$I_1 = \frac{2}{Z(\delta)} \int_{\mathbb{X}} \left(\exp(-\frac{1}{2}\Phi(W; \delta)) - \exp(-\frac{1}{2}\Phi(W; \delta')) \right) d\rho(W),$$

and

$$I_2 = 2|Z(\delta)^{-1/2} - Z(\delta')^{-1/2}|^2 \int_{\mathbb{X}} \exp(-\Phi(W; \delta')) d\rho(W).$$

From this, we deduce that $d_{\text{Hell}}(\rho^\delta, \rho^{\delta'}) \leq c|\delta - \delta'|$.

□

4 Inferring The Initial Condition

In the previous section we discussed the problem of inferring the forcing from the velocity field. In practical applications it is of interest to infer the initial condition, which corresponds to a Bayesian interpretation of 4DVAR, or the initial condition and the forcing, which corresponds to a Bayesian interpretation of weak constraint 4DVAR. Thus we consider the Bayesian inverse problem for inferring the initial condition u_0 and the white noise forcing determined by the Brownian driver W . Let ϱ be a Gaussian measure on the space \mathbb{H} and let $\mu = \varrho \otimes \rho$ be the prior probability measure on the space $\mathcal{H} = \mathbb{H} \times \mathbb{X}$. We denote the solution u of (3) by $u(x, t; u_0, W)$.

We outline what is required to extend the analysis of the previous two sections to the case of inferring both initial condition and driving Brownian motion. We simplify the presentation by assuming observation at only one

time $t_0 > 0$ although this is easily relaxed. Given that at $t_0 \in (0, T)$, the noisy observation δ of $\ell(u(\cdot, t_0; u_0, W))$ is given by

$$\delta = \ell(u(\cdot, t_0; u_0, W)) + \vartheta \quad (26)$$

where $\vartheta \sim N(0, \Sigma)$. Letting

$$\Phi(u_0, W; \delta) = \frac{1}{2} \|\delta - \ell(u(\cdot, t_0; u_0, W))\|_{\Sigma}^2, \quad (27)$$

we aim to show that the conditional probability $\mu^\delta = \mathbb{P}(u_0, W | \delta)$ is given by

$$\frac{d\mu^\delta}{d\mu} \propto \exp(-\Phi(u_0, W; \delta)). \quad (28)$$

We have the following result.

Theorem 3 *The conditional probability measure $\mu^\delta = \mathbb{P}(u_0, W | \delta)$ is absolutely continuous with respect to the prior probability measure μ with the Radon-Nikodym derivative given by (28). Further, there is a constant c such that*

$$d_{\text{Hell}}(\mu^\delta, \mu^{\delta'}) \leq c |\delta - \delta'|.$$

Proof We only sketch the proof because details are similar to those in the previous section. The key issue is establishing continuity of the forward map with respect to initial condition and driving Brownian motion. We show that $u(\cdot, t; u_0, W)$ is a continuous map from \mathcal{H} to \mathbb{H} . For $W \in \mathbb{X}$ and $u_0 \in \mathbb{H}$, we consider the following equation:

$$\frac{dv}{dt} + Av + B(v + z, v + z) = 0, \quad v(0) = u_0. \quad (29)$$

We denote the solution by $v(t) = v(t; u_0, W)$ to emphasize the dependence on initial condition and forcing which is important here. For $(u_0, W) \in \mathcal{H}$ and $(u'_0, W') \in \mathcal{H}$, from (16) and Gronwall's inequality, we deduce that

$$\begin{aligned} \|v(t; u_0, W) - v(t; u'_0, W')\|_{\mathbb{H}}^2 &\leq \|u_0 - u'_0\|_{\mathbb{H}}^2 e^{\int_0^t I_4(s') ds'} \\ &+ c \int_0^t \left(e^{\int_s^t I_4(s') ds'} \cdot \|z(s; W) - z(s; W')\|_{L^4(D)}^2 \cdot I_2(s) \right) ds. \end{aligned}$$

We then deduce that

$$\begin{aligned} \|v(t; u_0, W) - v(t; u'_0, W')\|_{\mathbb{H}}^2 &\leq c \|u_0 - u'_0\|_{\mathbb{H}}^2 + c \sup_{0 \leq s \leq T} \|z(s; W) - z(s; W')\|_{L^4(D)}^2 \\ &\leq c \|u_0 - u'_0\|_{\mathbb{H}}^2 + c \sup_{t \in (0, T)} \|W(t) - W'(t)\|_{\mathbb{H}^{1/2+\epsilon}}. \end{aligned}$$

This gives the desired continuity of the forward map.

For the Lipschitz dependency of the Hellinger distance of μ^δ on δ , we use the result of Mattingly [14] which shows that, for each initial condition u_0 ,

$$\mathbb{E}^\rho(\|u(t; u_0, W)\|_{\mathbb{H}}^2) \leq \frac{\mathcal{E}_0}{2\nu\lambda_1} + e^{-2\nu\lambda_1 t} (\|u_0\|_{\mathbb{H}}^2 - \frac{\mathcal{E}_0}{2\nu\lambda_1}),$$

where $\mathcal{E}_0 = \sum_{k=1}^{\infty} \sigma_k^2$. Therefore $\mathbb{E}^\mu(\|u(t; u_0, W)\|_{\mathbb{H}}^2)$ is bounded. The remainder of the proof follows as in Theorem 2. \square

5 Numerical Results

The purpose of this section is to demonstrate that the Bayesian formulation of the inverse problem described in this paper forms the basis for practical numerical inversion. In particular we show that it is possible to recover white noise forcing of the Navier-Stokes equation from linear functionals of the velocity field. In subsection 5.1 we describe the numerical method used for the forward problem. In subsection 5.2 we describe the inverse problem and the Metropolis-Hastings MCMC method used to probe the posterior. Subsection 5.3 describes the numerical results.

5.1 Forward Problem: Numerical Discretization

All our numerical results are computed using a viscosity of $\nu = 0.1$ and on the periodic domain. We work on the time interval $t \in [0, 0.1]$. We use $M = 32^2$ divergence free Fourier basis functions for a spectral Galerkin spatial approximation, and employ a time-step $\delta t = 0.01$ in a Taylor time-approximation [6]. The number of basis functions and time-step lead to a fully-resolved numerical simulation at this value of ν .

5.2 Inverse Problem: Metropolis Hastings MCMC

We consider the inverse problem of finding the driving Brownian motion. As a prior we take a centered Brownian motion in time with spatial covariance $\pi^4 A^{-2}$; thus the space-time covariance of the process is $C_0 := \pi^4 A^{-2} \otimes \Delta_t^{-1}$, where Δ_t is the Laplacian in time with fixed homogeneous Dirichlet condition at $t = 0$ and homogeneous Neumann condition at $t = T$. It is straightforward to draw samples from this Gaussian measure, using the fact that A is diagonalized in the spectral basis. Note that if $W \sim \rho$, then $W \in C(0, T; \mathbb{H}^s)$ almost surely for all $s < 1$; in particular $W \in \mathbb{X}$. Thus $\rho(\mathbb{X}) = 1$ as required. The likelihood is defined by making observations of the velocity field at every point on the 32^2 grid implied by the spectral method, at every time $t = n\delta t$, $n = 1, \dots, 10$. The observational noise standard deviation is taken to be $\gamma = 3.2$.

To sample from the posterior distribution we employ a Metropolis-Hastings MCMC method. Furthermore, to ensure mesh-independent convergence properties, we use a method which is well-defined in Hilbert space [1]. Metropolis-Hastings methods proceed by constructing a Markov kernel \mathcal{P} which satisfies *detailed balance* with respect to the measure ρ^δ which we wish to sample:

$$\rho^\delta(du)\mathcal{P}(u, dv) = \rho^\delta(dv)\mathcal{P}(v, du), \quad \forall u, v \in \mathbb{X}. \quad (30)$$

Integrating with respect to u , one can see that detailed balance implies $\rho^\delta \mathcal{P} = \rho^\delta$. Metropolis-Hastings methods [5, 16] prescribe an accept-reject move based

on proposals from another Markov kernel \mathcal{Q} , in order to define a kernel \mathcal{P} which satisfies detailed balance. If we define the measures

$$\begin{aligned}\nu(du, dv) &= \mathcal{Q}(u, dv)\rho^\delta(du) \propto \mathcal{Q}(u, dv) \exp(-\Phi(u; \delta))\rho(du) \\ \nu^\perp(du, dv) &= \mathcal{Q}(v, du)\rho^\delta(dv) \propto \mathcal{Q}(v, du) \exp(-\Phi(v; \delta))\rho(dv).\end{aligned}\quad (31)$$

then, provided $\nu^\perp \ll \nu$, the Metropolis-Hastings method is defined as follows. Given current state u_n , a proposal is drawn $u^* \sim \mathcal{Q}(u_n, \cdot)$, and then accepted with probability

$$\alpha(u_n, u^*) = \min \left\{ 1, \frac{d\nu^\perp}{d\nu}(u_n, u^*) \right\}. \quad (32)$$

The resulting chain is denoted by \mathcal{P} . If the proposal \mathcal{Q} preserves the prior, so that $\rho\mathcal{Q} = \rho$, then a short calculation reveals that

$$\alpha(u_n, u^*) = \min \left\{ 1, \exp(\Phi(u_n; \delta) - \Phi(u^*; \delta)) \right\}; \quad (33)$$

thus the acceptance probability is determined by the change in the likelihood in moving from current to proposed state. We use the following pCN proposal [1] which is reversible with respect to the Gaussian prior

$$\mathcal{Q}(u_n, \cdot) = \mathcal{N}(\sqrt{1 - \beta^2}u_n, \beta^2 C_0) \quad (34)$$

and thus results in the acceptance probability (33). Variants on this algorithm, which propose differently in different Fourier components, are described in [12], and can make substantial speedups in the Markov chain convergence. However for the examples considered here the basic form of the method suffices.

5.3 Results and Discussion

The true driving Brownian motion W^\dagger , underlying the data in the likelihood, is constructed as a draw from the prior ρ . We then compute the corresponding true trajectory $u^\dagger(t) = u(t; W^\dagger)$. We use the pCN scheme (33),(34) to sample W from the posterior distribution ρ^δ . The true initial and final conditions are plotted in Figure 1, top two panels, for the vorticity field w ; the bottom two panels of Figure 1 show the posterior mean of the same quantities and indicate that the data is fairly informative, since they closely resemble the truth. The true trajectory, together with the posterior mean and one standard deviation interval around the mean, are plotted in Figure 2, for the wavenumbers $(0, 1)$ and $(4, 4)$, and for both the driving Brownian motion W and the velocity field u . This figure indicates that the data is very informative about the $(0, 1)$, but less so concerning the $(4, 4)$ mode for which the mean and standard deviation exhibit behaviour similar to that under the prior.

Acknowledgements VHH is supported by a start up grant from Nanyang Technological University, AMS is grateful to EPSRC, ERC, ESA and ONR for financial support for this work, and KJHL is grateful to ESA for financial support.

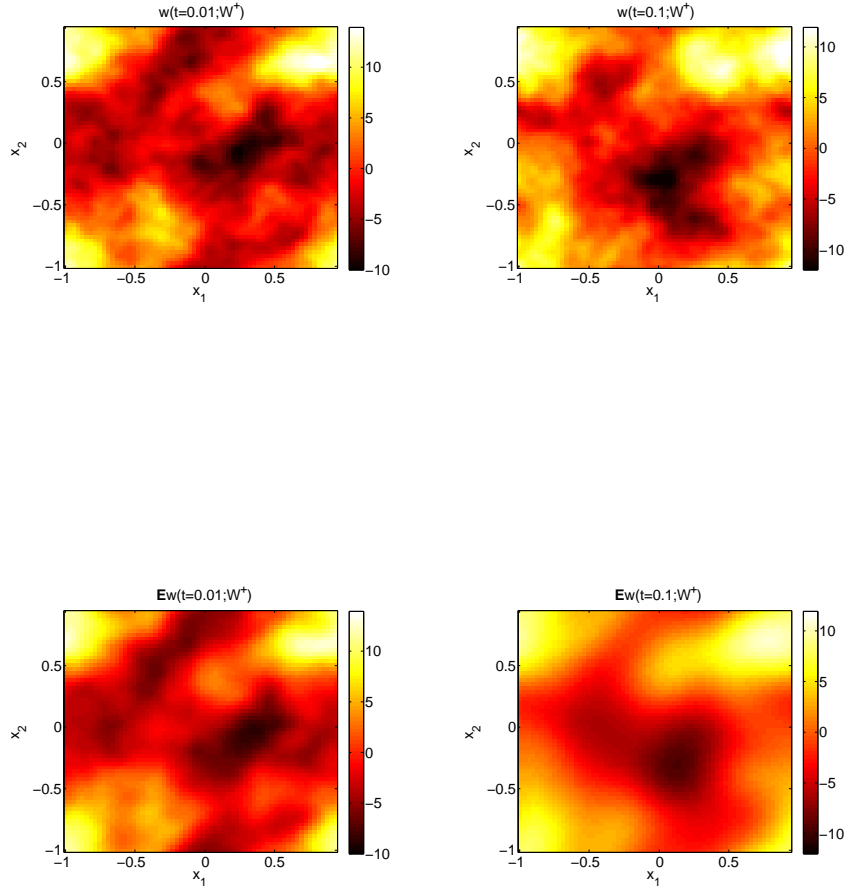


Fig. 1 The truth (top) and expected value (bottom) of $u(t; W)$, with $t = 0.01$ (left) and $t = 0.1$ (right).

References

1. S. Cotter, G. Roberts, A. Stuart, and D. White. MCMC methods for functions: modifying old algorithms to make them faster. *Arxiv preprint arXiv:1202.0709*, 2012.
2. S. L. Cotter, M. Dashti, J. C. Robinson, and A. M. Stuart. Bayesian inverse problems for functions and applications to fluid mechanics. *Inverse Problems*, 25:115008, 2009.

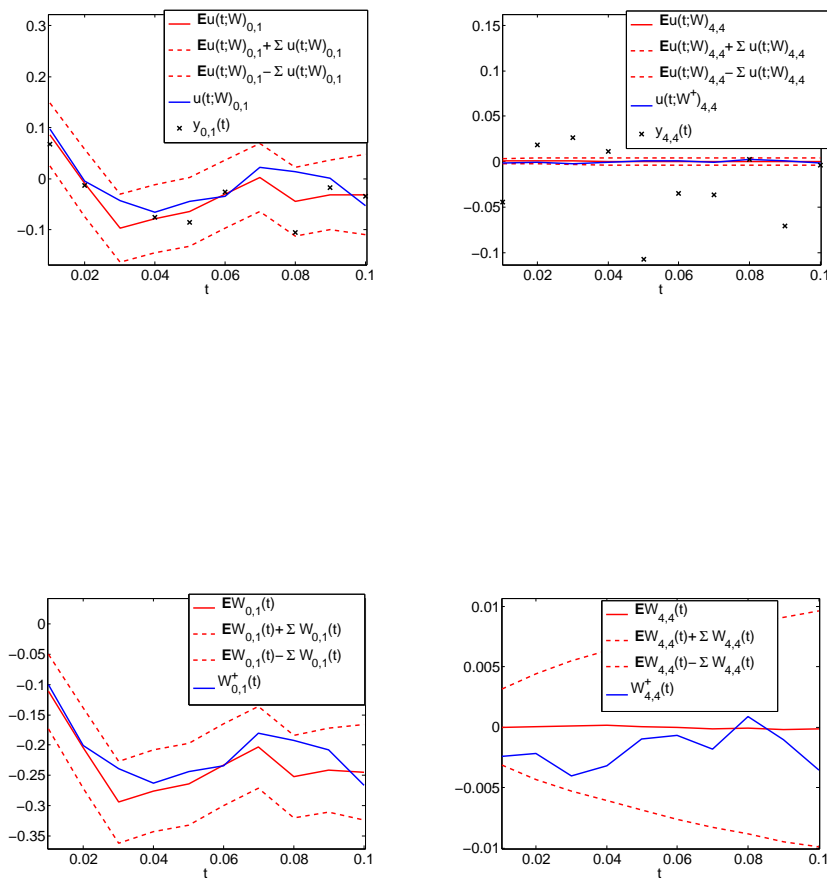


Fig. 2 The trajectories $u_k(t; W)$ (top) and W_k (bottom), with $k = (0, 1)$ (left) and $k = (4, 4)$ (right). Shown are expected values and standard deviation intervals as well as true values.

3. Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, 2008.
4. F. Flandoli. Dissipative and invariant measures for stochastic Navier-Stokes equations. *NODEA*, 1:403–423, 1994.
5. W.K. Hastings. Monte carlo sampling methods using markov chains and their applications. *Biometrika*, 57:97–109, 1970.

6. A. Jentzen and P. Kloeden. Taylor expansions of solutions of stochastic partial differential equations with additive noise. *The Annals of Probability*, 38(2):532–569, 2010.
7. Jari Kaipio and Erkki Somersalo. *Statistical and computational inverse problems*, volume 160. Springer, 2004.
8. S. Lasanen. Discretizations of generalized random variables with applications to inverse problems. *Ann. Acad. Sci. Fenn. Math. Diss., University of Oulu*, 130, 2002.
9. S. Lasanen. Measurements and infinite-dimensional statistical inverse theory. *PAMM*, 7:1080101–1080102, 2007.
10. Sari Lasanen. Non-gaussian statistical inverse problems. part i: Posterior distributions. *Inverse Problems and Imaging*, 6(2):215–266, 2012.
11. Sari Lasanen. Non-gaussian statistical inverse problems. part ii: Posterior convergence for approximated unknowns. *Inverse Problems and Imaging*, 6(2):267–287, 2012.
12. K.J.H. Law. Proposals which speed-up function space MCMC. *arXiv preprint arXiv:1212.4767*, 2012.
13. Andrew C Lorenc. The potential of the ensemble kalman filter for nwp comparison with 4d-var. *Quarterly Journal of the Royal Meteorological Society*, 129(595):3183–3203, 2003.
14. J. C. Mattingly. Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity. *Commun. Math. Phys.*, 206:273–288, 1999.
15. Andrew M Stuart. Inverse problems: a bayesian perspective. *Acta Numerica*, 19(1):451–559, 2010.
16. L. Tierney. A note on Metropolis-Hastings kernels for general state spaces. *Ann. Appl. Probab.*, 8(1):1–9, 1998.
17. Sebastian J Vollmer. Dimension-independent MCMC sampling for elliptic inverse problems with non-Gaussian priors. *arXiv preprint arXiv:1302.2213*, 2013.
18. Dusanka Zupanski. A general weak constraint applicable to operational 4dvar data assimilation systems. *Monthly Weather Review*, 125(9):2274–2292, 1997.